distributed computer systems

Rheinland-Pfälzische Technische Universität Kaiserslautern Landau

Using Minimal Arrival Curves to Derive Per-Flow Performance Bounds in (Tandem) Networks with Complex Feedback Structures [1]

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[1] to be published in ECRTS 2025

Outline

1 Previous NC Work on Feedback

2 Extending Analysis to Interdependent Feedback Constraints

3 Deriving Per-Flow Performance Bounds

4 Conclusion

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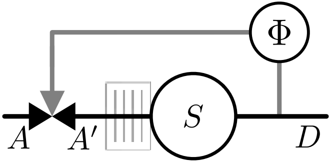

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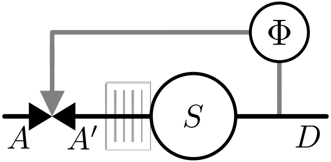

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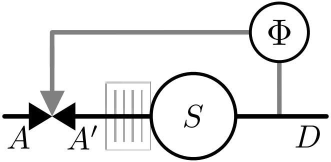

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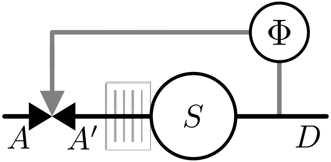

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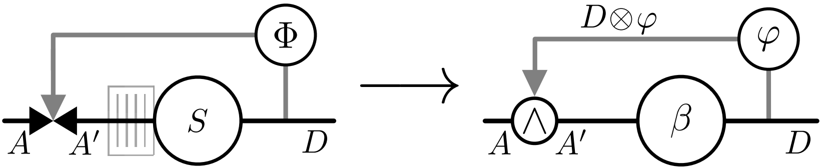

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- Signaling is modeled as backwards loop that feeds back into dataflow
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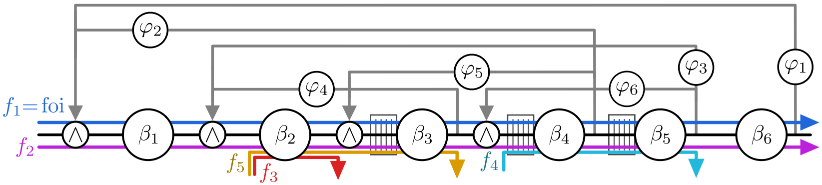
- Arrivals to the system are throttled according to this function
- Call φ with $D \circ \Phi \geq D \otimes \varphi$ a feedback curve

What Do We Want To Analyze?



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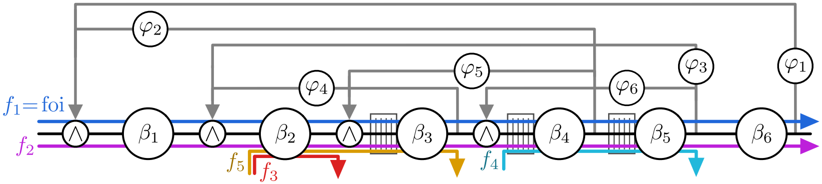

Figure: Example network with various feedback constraints and flows.

- Network with a multitude of feedback constraints that potentially have interdependencies
- Arbitrary number of interfering flows
- Goal: Obtain per-flow performance bounds for any scheduling policy
  - Not just FIFO...

Closed-Loop System

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- Solution: transformation into *open-loop system* [Chang, 2000]

Theorem

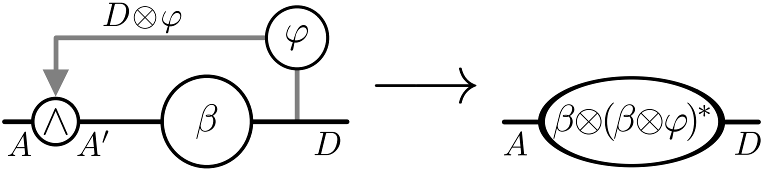
Let an arrival process A traverse a system $\mathcal{S}$ offering service curve $\beta \in \mathcal{F}$. The system is subject to a feedback curve φ. If

$$D \geq (A \wedge (D \otimes \varphi)) \otimes \beta,$$

then

$$D \geq A \otimes \beta \otimes (\beta \otimes \varphi)^{*}.$$

Transformation Into Open-Loop System



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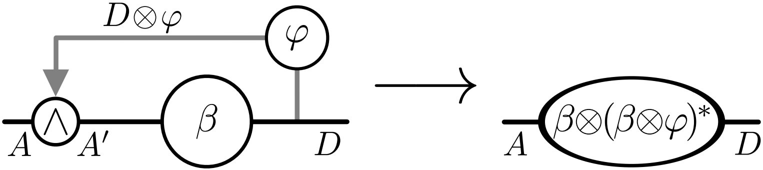

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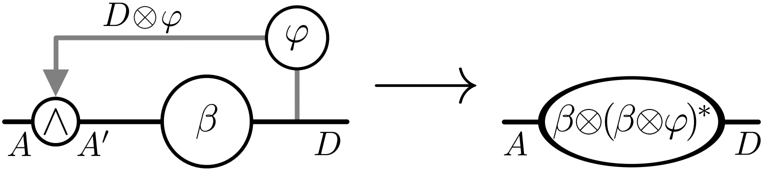

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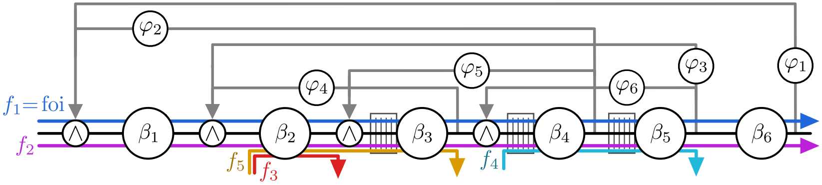
- Feedback is still captured in the system, but encoded into the service curve directly
- Resulting system is min-plus linear → can be analyzed with normal NC methods

Generalization To Arbitrary Length



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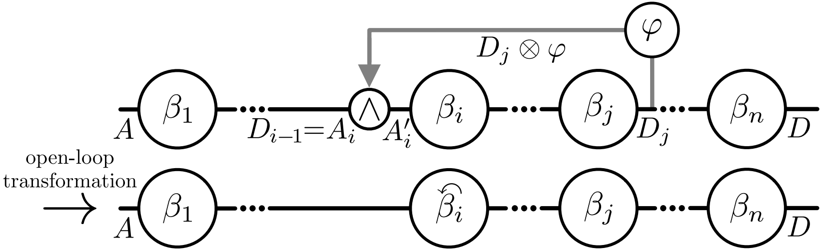

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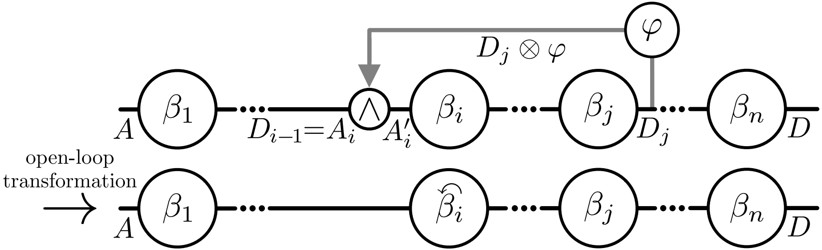

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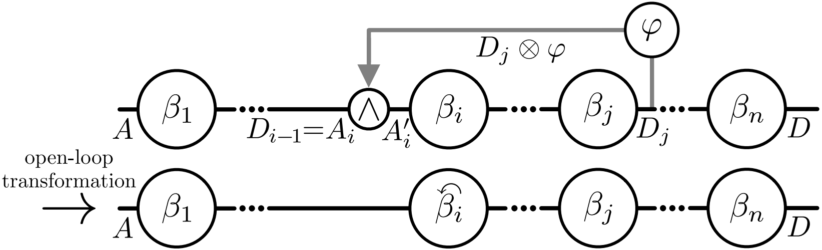

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- Call this Proposition for remainder of the talk

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- Some existing results on overlapping interdependency [Chang, 2000, Bose et al., 2006, Bouillard et al., 2009]

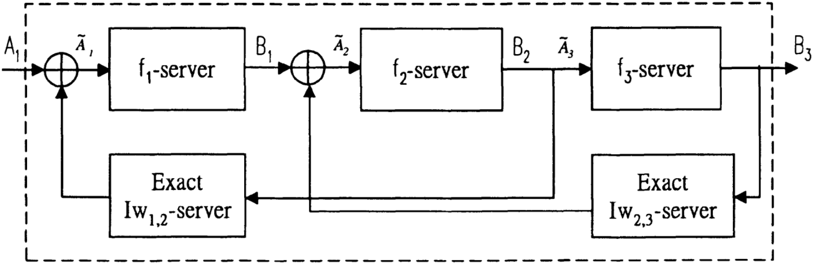

Figure: Overlapping interdependency [Chang, 2000].

Complex Feedback Structures

- What about having multiple constraints in a single system?
- What if the constraints have an interdependency?
- Some existing results on overlapping interdependency [Chang, 2000, Bose et al., 2006, Bouillard et al., 2009]
- But, not the only interdependence type!

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- Establish lexicographical order over all existing feedback constraints $F_l = (i_l, j_l, \varphi_l), l = 1, \ldots, k$, in the system as follows

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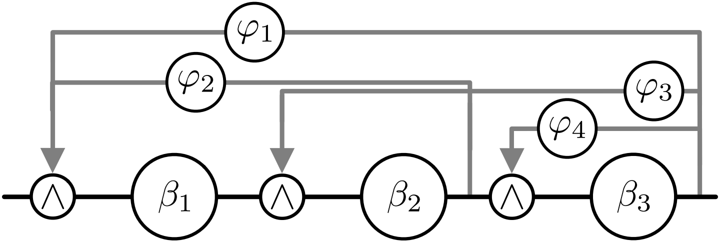

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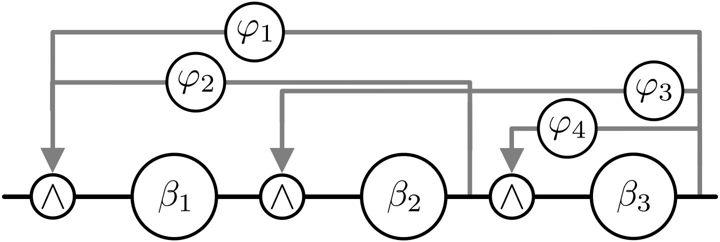

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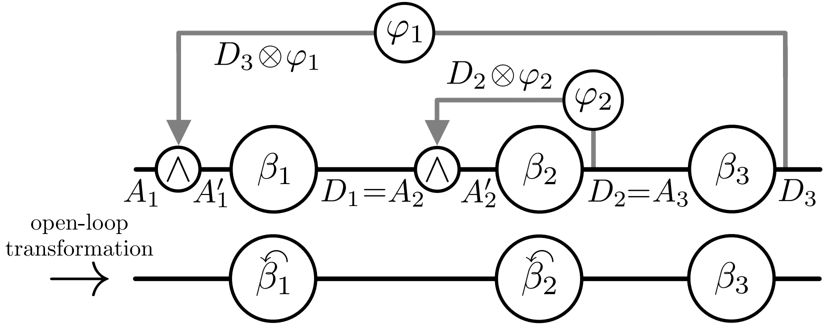

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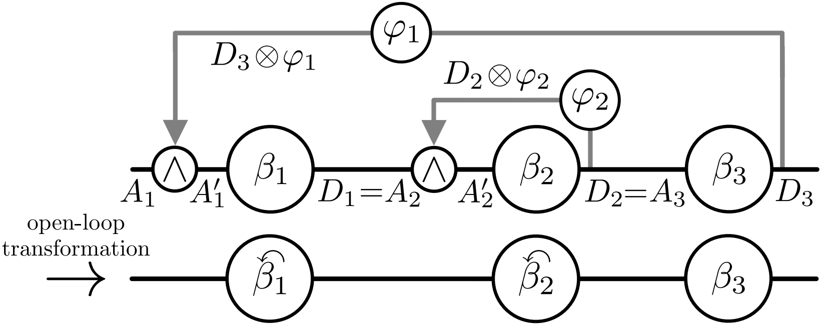

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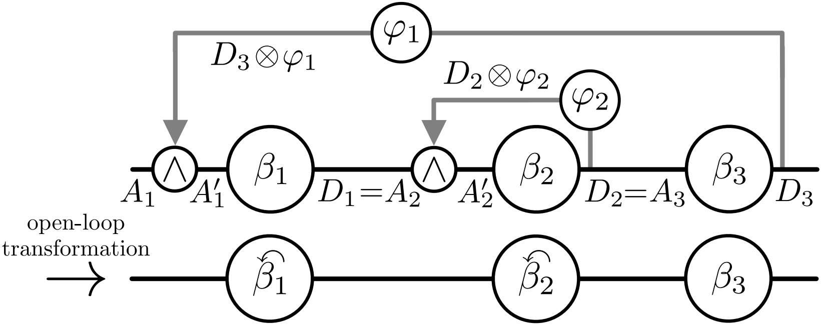

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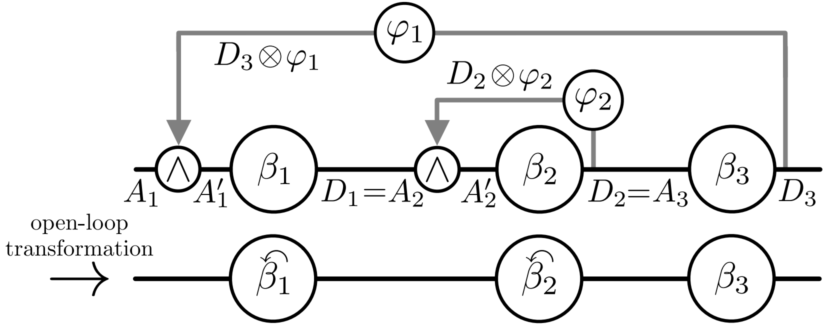

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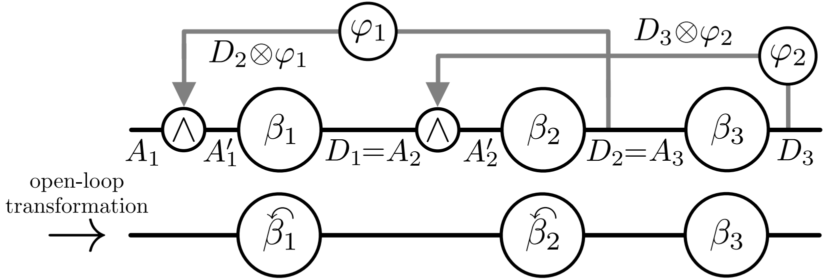

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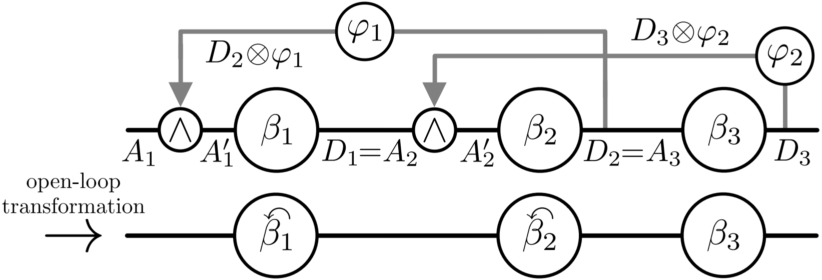

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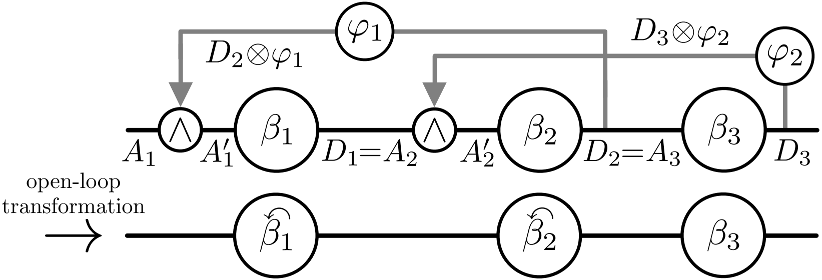

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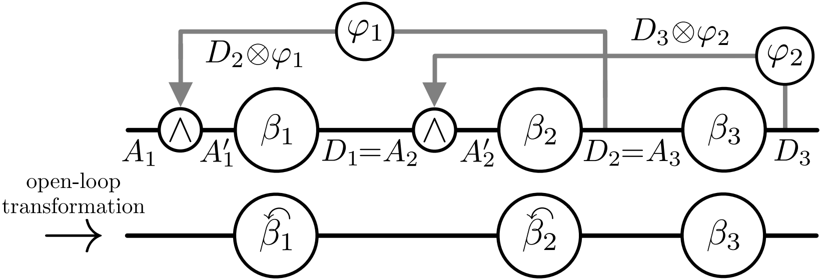

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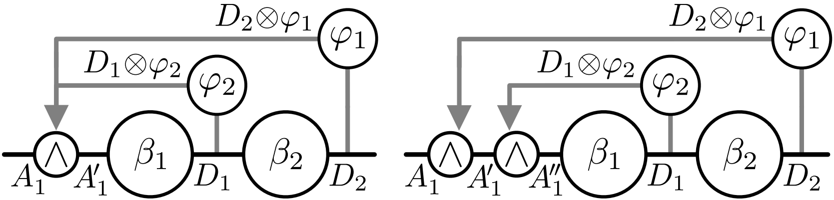

(a) Combined minimum

(b) Separated minima

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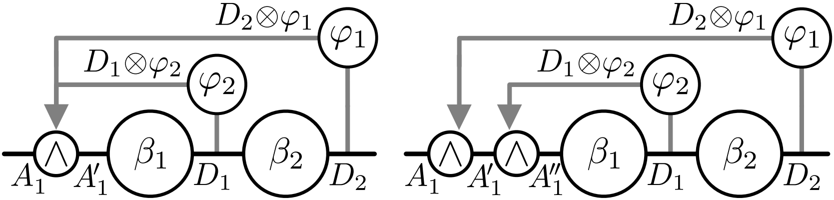

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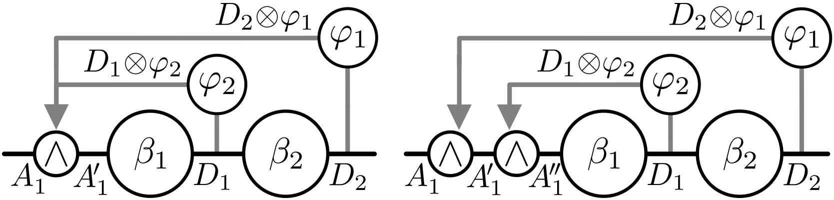

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- Can show that structure (a) is equivalent to structure (b)
  - (a) $A'_1 = A_1 \wedge (D_2 \otimes \varphi_1) \wedge (D_1 \otimes \varphi_2)$
  - (b) $A'_1 = A_1 \wedge (D_2 \otimes \varphi_1)$,
    $A''_1 = A'_1 \wedge (D_1 \otimes \varphi_2) = A_1 \wedge (D_2 \otimes \varphi_1) \wedge (D_1 \otimes \varphi_2)$

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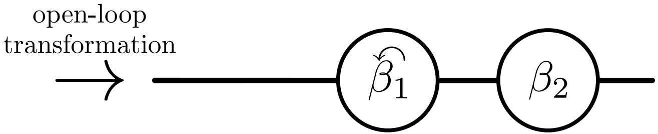

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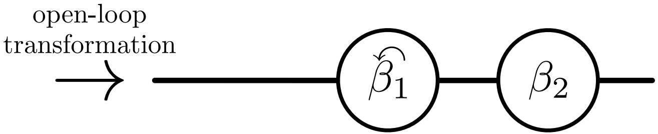

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- Have seen in all canonical example that there is one feedback constraint that can be safely resolved first
- Using the established ordering of feedback constraints, this is always the feedback with the largest ordering index
- General procedure:
  1. Transform feedback constraint F_l with largest index l and exchange β_{i_l} with $\overset{\curvearrowleft}{\beta}_{i_l}$
  2. Remove F_l from set of feedback constraints
  3. Repeat until no feedback constraints are left in the system

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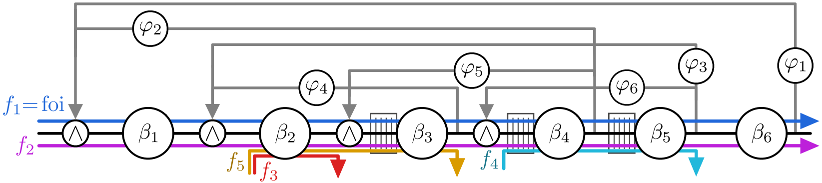

Figure: Example network before transformation.

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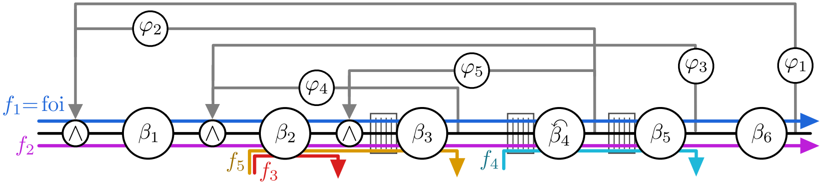

Figure: First transformation.

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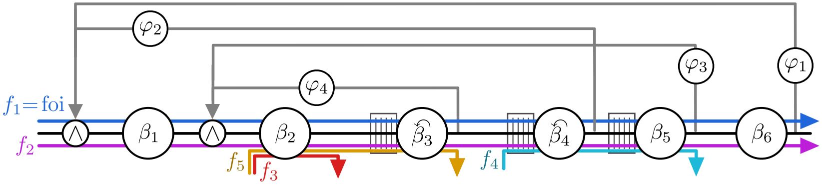

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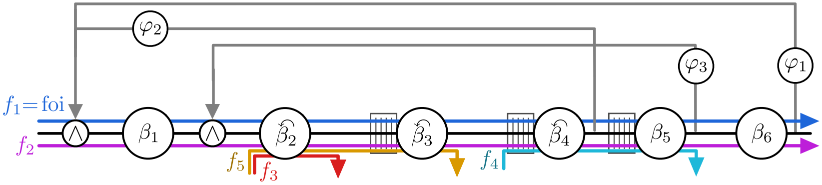

Figure: Third transformation.

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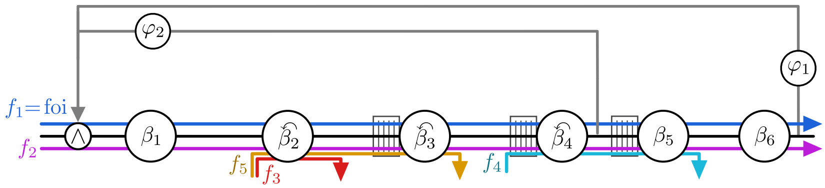

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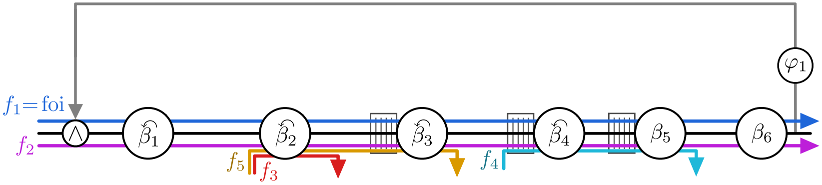

Figure: Fifth transformation.

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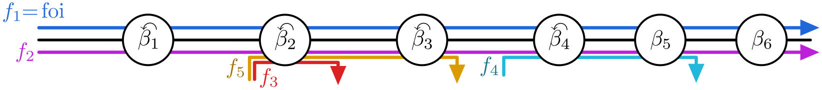

Figure: Transformed open-loop system.

Deriving Per-Flow Performance Bounds

1 Previous NC Work on Feedback

2 Extending Analysis to Interdependent Feedback Constraints

3 Deriving Per-Flow Performance Bounds

4 Conclusion

Obtaining A Residual Service Curve

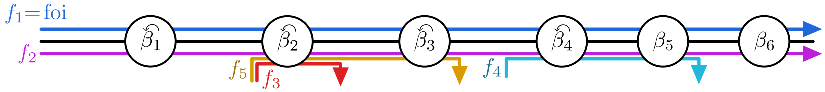

Figure: Open-loop system of the introductory example.

- Start with an open-loop system, where each transformed node offers a *min-plus* service curve

Obtaining A Residual Service Curve

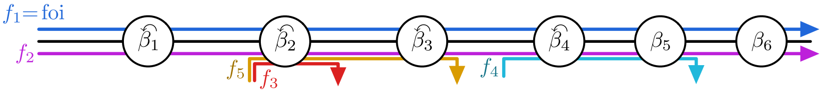

Figure: Open-loop system of the introductory example.

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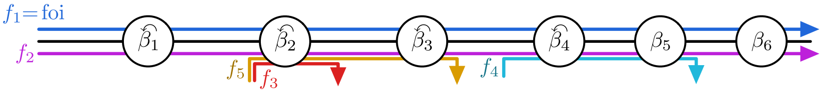

Figure: Open-loop system of the introductory example.

- Start with an open-loop system, where each transformed node offers a *min-plus* service curve
- Interested in performance bounds for a flow of interest f_1
- Use general residual service curve formula from [Bouillard et al., 2018]

$$\beta_i^{\mathrm{res}} = \beta_i - \overline{\alpha}_i^{\mathrm{c}}$$

with $\overline{\alpha}_i^c$ the maximal arrival curve for the cross flows

Obtaining Per-Flow Performance Bounds

- Use result from [Hamscher et al., 2024] to obtain performance bounds

Theorem

Let an arrival process A traverse a system $\mathcal{S}$. Further, let the arrivals be constrained by maximal arrival curve $\overline{\alpha} \in \mathcal{F}_0^{\uparrow}$, and minimal arrival curve $\underline{\alpha} \in \mathcal{F}_0^{\uparrow}$, and let the system offer a min-plus service curve $\beta \in \mathcal{F}$, with $\xi := \beta_{\downarrow}$ its lower non-decreasing closure. The backlog $q(t)$ satisfies for all t

$$q(t) \leq v(\overline{\alpha}, \xi) \wedge \sup_{s \geq 0}\{\overline{\alpha}(s)\}.$$

The virtual delay $d(t)$ satisfies for all $t \geq 0$

$$d(t) \leq \max\{z(\underline{\alpha}, \xi), h(\overline{\alpha}, \xi)\}.$$

with $z(\underline{\alpha}, \xi) := \inf\{\tau \geq 0 | \underline{\alpha} \otimes \xi(\tau) \geq 0\}$

Conclusion

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$\Rightarrow$ Transform feedback constraints in the order provided by the topological order

- Applying this method to tandem networks provides the correct order of evaluation

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Any questions?

Generalization To Arbitrary Length

- Can generalize the Theorem to transform a system with a feedback constraint of arbitrary length

Proposition

Let a flow f traverse a tandem of n nodes, each offering a service curve $\beta_k, k \in \{1, \ldots, n\}$, with a single feedback constraint $F = (i, j, \varphi)$. The tandem offers an end-to-end service curve to the flow f

$$D \geq A \otimes \beta_{\mathrm{e2e}}, \text{ with } \beta_{\mathrm{e2e}} := \bigotimes_{k=1}^{i-1} \beta_k \otimes \overset{\curvearrowleft}{\beta_i} \otimes \bigotimes_{l=i+1}^{n} \beta_l, \tag{1}$$

and a transformed service curve at node i

$$\overset{\curvearrowleft}{\beta_i} := \beta_i \otimes \left(\bigotimes_{k=i}^{j} \beta_k \otimes \varphi \right)^{*}. \tag{2}$$

Compounded Interdependency

Note that in the case of $i_k = i_l$ and $j_k = j_l$, i.e., the extreme case of compounded feedback spanning the same set of nodes, we only need to evaluate a single feedback loop, however, with feedback constraint $\varphi_k \wedge \varphi_l$. Using distributivity,

$$A' = A \wedge (D \otimes \varphi_k) \wedge (D \otimes \varphi_l) = A \wedge D \otimes (\varphi_k \wedge \varphi_l).$$